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Note

Small Embeddings of Partial Directed Triple Systems and Partial Triple Systems with Even λ

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In 1976, Lindner and Rosa (*Ars Combin.* **1** (1976), 159–166) showed that a partial triple system with $\lambda > 1$ can be embedded in a finite triple system with the same λ. This result is improved in the case when λ is even by embedding a partial triple system on v symbols in a triple system on t symbols, $t \equiv 0, 1 \pmod 3$, for all $t \geqslant 3(\lambda v^2 + v(2-\lambda) + 1)$. In the process, it is shown that for any $\lambda \geqslant 1$, a partial directed triple system on v symbols can be embedded in a directed triple system on t symbols, $t \equiv 0, 1 \pmod 3$, for all $t \geqslant 6\lambda v^2 + 6v(1-\lambda) + 3$, thus generalizing a result of Hamm (Proceedings, 14th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Boca Raton, Florida, 1983). © 1984 Academic Press, Inc.

1. INTRODUCTION

A *block design* $B[k, \lambda; v]$ is a pair (V, B), where V is a v-set of symbols and B is a collection of k-subsets of V called *blocks*, such that each unor-

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dered pair of symbols appears in precisely λ blocks. A *directed block design* $DB[k, \lambda; v]$ is a pair (V, B), where B is a collection of blocks which are k-tuples of symbols from V. A block contains an ordered pair (x, y) if x precedes y in the tuple; each ordered pair is contained in precisely λ blocks. Directed block designs have also been studied under the name *transitive block designs*. A *Mendelsohn block design* $MB[k, \lambda; v]$ is a pair (V, B), where B contains the ordered pair (x, y) if x precedes y and is adjacent to y, and the last symbol in any block is adjacent to, and precedes, the first; again, each ordered pair is contained in precisely λ blocks. Mendelsohn designs have also been investigated under the name *cyclic block designs*, and hence blocks are often called *cyclic blocks* in this case. When $k = 3$, these various types of designs are referred to as *triple systems*. In each case, a *partial* design is one in which the constraints are relaxed to allow (ordered) pairs to appear at most λ times, rather than precisely λ times.

Hanani [10] has shown that triple systems $B[3, \lambda; v]$ exist if and only if $\lambda v(v-1) \equiv 0 \pmod 6$; therefore, triple systems $B[3, 2\lambda; v]$ exist if and only if $v \equiv 0, 1 \pmod 3$. The embedding problem for partial triple systems was first examined in the case $\lambda = 1$ by Treash [18], who established that partial triple systems $B[3, 1; v]$ have finite embeddings. Lindner [12, 13] subsequently substantially reduced the size of the containing system; to date, the best result is that a partial triple system $B[3, 1; v]$ can be embedded in a triple system $B[3, 1; w]$ for every $w \geqslant 4v + 1$, $w \equiv 1, 3 \pmod 6$ [1]. Determining the size of the smallest containing system has been shown to be *NP*-hard [4], and thus a good characterization of partial triple systems with very small embeddings is unlikely.

In the case of arbitrary λ, Lindner and Rosa established the following

THEOREM 1 [14]. *A partial triple system $B[3, \lambda; v]$ can be embedded in a triple system $B[3, \lambda; t]$ for some finite integer t.*

In this paper, we investigate how small such an embedding can be. In particular, we establish

THEOREM 2. *Let S be a partial triple system $B[3, 2\lambda; v]$. S can be embedded in a triple system $B[3, 2\lambda; t]$ for any $t \geqslant 6(|E(S)| + v) + 3$, $t \equiv 0, 1$* (mod 3), *where $|E(S)|$ is the number of pairs* (*counting repetitions*) *occurring in blocks of S.*

COROLLARY 3. *A partial triple system $B[3, 2\lambda; v]$ can be embedded in a triple system $B[3, 2\lambda; t]$ for any $t \geqslant 6(\lambda v^2 + v(1-\lambda)) + 3$, $t \equiv 0, 1$* (mod 3).

Proof of Corollary. The number of pairs on v symbols is at most $v(v-1)/2$. Since each pair appears in at most 2λ blocks, $|E(S)| \leqslant \lambda v(v-1)$.

2. Directed Triple Systems

Hung and Mendelsohn [11] first studied directed block designs, showing that a $DB[3, 1; v]$ exists for every $v \equiv 0, 1 \pmod 3$. Necessary and sufficient conditions for the existence of a $DB[3, \lambda; v]$ have since been found for $k = 3$ [15], $k = 4$ [16], and $k = 5$ [17]. The relation between directed triple systems and (undirected) triple systems is quite close. A partial triple system $B[3, 2\lambda; v]$ P is said to *underlie* a partial directed triple system $DB[3, \lambda; v]$ D when P is obtained from D by ignoring the ordering on triples. It is easy to see that every partial directed triple system has an underlying triple system; the converse also holds:

THEOREM 4 [5, 7]. *Every triple system $B[3, 2\lambda; v]$ underlies some directed triple system $DB[3, \lambda; v]$.*

COROLLARY 5. *Every partial triple system $B[3, 2\lambda; v]$ is the underlying system of some partial directed triple system $DB[3, \lambda; v]$.*

Proof of Corollary. Using Theorem 1, embed the partial triple system P finitely in a triple system T. Using Theorem 4, produce a directed triple system D whose underlying system is T. Restricting attention to those triples of D arising from P produces a partial directed triple system which has P as its underlying partial triple system.

Corollary 5 enables us to transform embedding questions about partial triple systems into embedding questions about partial directed triple systems. We therefore consider the problem of embedding a partial directed triple system $DB[3, \lambda; v]$; the case $\lambda = 1$ has been previously studied by Hamm [8]. In the more general case, we establish

THEOREM 6. *A partial directed triple system $DB[3, \lambda; v]$ **SD** can be embedded in a directed triple system $DB[3, \lambda; t]$ for every $t \geqslant 6(|E(\mathbf{SD})| + v) + 3$, $t \equiv 0, 1 \pmod 3$, where $|E(\mathbf{SD})|$ is the number of ordered pairs (counting repeated pairs) appearing in blocks of* **SD**.

COROLLARY 7. *A partially directed triple system $DB[3, \lambda; v]$ can be embedded in a directed triple system $DB[3, \lambda; t]$ for every $t \geqslant 6(\lambda v^2 + v(1 - \lambda)) + 3$, $t \equiv 0, 1 \pmod 3$.*

Theorem 2 follows as an immediate consequence of Theorem 6 and Corollary 5; hence in the remainder of the paper we develop the proof of Theorem 6.

3. Embedding Partial Directed Triple Systems

The development of the small embedding of partial directed triple systems requires two known results and a lemma as background. The first result required is a well-known colouring theorem due to Brooks:

THEOREM 8 [3]. *A graph with maximum degree $\lambda - 1$ can be properly vertex coloured with λ colours.*

THEOREM 9 [2]. *A partial idempotent quasigroup of size n can be embedded in an idempotent quasigroup of size t for every $t \geqslant 2n+1$.*

LEMMA 10. *Let* **PD** *be a partial directed triple system $DB[3, \lambda; n]$ and let* **PM** *be a partial Mendelsohn triple system $MB[3, \lambda; n]$. Furthermore, suppose that*

(1) *each ordered pair appears the same number of times in* **PD** *and in* **PM**—*this is referred to as* mutual balance;

(2) *the cyclic triples of* **PM** *can be partitioned into λ sections so that two triples having an ordered pair in common occur in different sections.*

Then **PD** *can be embedded in a directed triple system $DB[3, \lambda; t]$ for all $t \geqslant 6n+3$, $t \equiv 0, 1 \pmod 3$.*

Proof. Let the n symbols **PM** and **PD** be $v_1, v_2, \ldots, v_n$, and suppose that **PM** contains r cyclic triples which we arbitrarily label $b_1, \ldots, b_r$. The postulate of mutual balance ensures that there is a bijection θ from the ordered pairs (with repetitions included) in **PD** to the ordered pairs in **PM** so that for all ordered pairs (x, y) in a block of **PD**, $\theta((x, y)) = (x, y)$.

Using postulate (2), partition the triples of **PM** into λ sections so that cyclic triples having an ordered pair in common are placed in different sections. Form λ different partial idempotent quasigroups $Q_1, \ldots, Q_\lambda$ of size n on the symbols $v_1, \ldots, v_n$ with products $\bigcirc_1, \ldots, \bigcirc_\lambda$ respectively, as follows: for $1 \leqslant i \leqslant \lambda$ and $1 \leqslant j \leqslant n$ define $v_j \bigcirc_i v_j = v_j$; for $1 \leqslant j \leqslant r$, if $b_j = (v_x, v_y, v_z)$ occurs in section i, then define $v_x \bigcirc_i v_y = v_z$, $v_y \bigcirc_i v_z = v_x$, and $v_z \bigcirc_i v_x = v_y$.

Theorem 9 ensures that each Q_i, $1 \leqslant i \leqslant \lambda$, can be embedded in an idempotent quasigroup of size $s = t/3$ on the symbols $v_1, \ldots, v_{t/3}$ if $t \equiv 0 \pmod 3$, and of size $s = (t-1)/3$ on the symbols $v_1, \ldots, v_{(t-1)/3}$ if $t \equiv 1 \pmod 3$, which is possible since $s \geqslant 2n+1$.

Then we can embed **PD** in a directed triple system $DB[3, \lambda; t]$ T on the symbols $\{(v_i, j) \mid 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant 3\}$ if $t \equiv 0 \pmod 3$, or on the same set together with ∞ if $t \equiv 1 \pmod 3$, by selecting directed blocks as follows:

(1) if $t \equiv 0 \pmod 3$, for each $1 \leqslant i \leqslant s$, λ copies of the directed blocks $((v_i, 1), (v_i, 2), v_i, 3))$ and $((v_i, 3), (v_i, 2), (v_i, 1))$ are included; if $t \equiv 1$

(mod 3), for each $1 \leqslant i \leqslant s$, λ copies of the directed blocks $(\infty, (v_i, 1), (v_i, 2))$, $((v_i, 3), (v_i, 2), \infty)$, $((v_i, 1), \infty, (v_i, 3))$, and $((v_i, 2), (v_i, 3), v_i, 1))$ are included;

(2) for each directed triple in **PD**, say (v_j, v_k, v_l), we include the directed blocks $((v_j, 1), (v_k, 1), (v_l, 1))$, $((v_j, 2), (v_k, 2), (v_l, 2))$, $((v_j, 3), (v_k, 3), (v_l, 3))$, $((v_l, 2), (v_k, 3), (v_j, 1))$, $((v_l, 3), (v_k, 2), (v_j, 1))$, $((v_l, 3), (v_k, 1), (v_j, 2))$, $((v_l, 1), (v_k, 3), (v_j, 2))$, $((v_l, 2), (v_k, 1), (v_j, 3))$, and $((v_l, 1), (v_k, 2), (v_j, 3))$;

(3) for $1 \leqslant i \leqslant \lambda$ and $1 \leqslant j, k, l \leqslant s$, if $v_j \bigcirc_i v_k = v_l$ and (v_j, v_k, v_l) is not a cyclic triple in **PM** occurring in section i, then the directed blocks $((v_j, 1), (v_l, 2), (v_k, 1))$, $((v_j, 2), (v_l, 3), (v_k, 2))$, and $((v_j, 3), (v_l, 1), (v_k, 3))$ are included.

The blocks in part (2) substitute the directed blocks of **PD** for the cyclic triples; this creates no difficulty since the sets of triples are mutually balanced. Then it can be verified that T is a directed triple system in which **PD** is embedded.

This lemma gives us the basic ingredient to find small embeddings of partial directed triple systems:

Proof of Theorem 6. Let **SD** be a partial directed triple system $DB[3, \lambda; v]$ containing s directed triples which we arbitrarily label $b_1, \dots, b_s$ and let the symbols of **SD** be $\alpha_1, \dots, \alpha_v$. As a first step, we embed **SD** in a partial directed triple system **PD** $DB[3, \lambda; 3s + v]$ in such a way that there exists a partial Mendelsohn triple system $MB[3, \lambda; 3s + v]$ **PM**, where **PM** and **PD** are mutually balanced, and **PM** has a partitioning into λ sections as required by Lemma 10. The application of Lemma 10 will then result in an embedding of **SD** in a directed triple system **TD**, establishing the result.

In order to obtain **PD**, we introduce $3s$ new vertices $\alpha_{v+1}, \dots, \alpha_{v+3s}$; for each directed triple $b_i = (\alpha_j, \alpha_k, \alpha_l)$ in **TD**, add the directed triples $(\alpha_{v+3i-1}, \alpha_k, \alpha_{v+3i-2})$, $(\alpha_{v+3i-2}, \alpha_{v+3i}, \alpha_j)$, and $(\alpha_l, \alpha_{v+3i}, \alpha_{3i-1})$ which we label $b_{i,k}$, $b_{i,j}$, and $b_{i,l}$, respectively. Notice that ordered pairs (α_i, α_j) appear at most once if either i or j exceeds v, when $i, j \leqslant v$, the ordered pair appears at most λ times.

We next construct a partial Mendelsohn triple system P_M which is in mutual balance with **PD**. For each directed triple $b_i = (\alpha_j, \alpha_k, \alpha_l)$ in **SD**, let **PM** contain the cyclic triples $(\alpha_j, \alpha_k, \alpha_{v+3i-2})$, $(\alpha_k, \alpha_l, \alpha_{v+3i-1})$, $(\alpha_j, \alpha_l, \alpha_{v+3i})$, and $(\alpha_{v+3i-2}, \alpha_{v+3i}, \alpha_{v+3i-1})$ which we label with $c_{i,1}, c_{i,2}, c_{i,3}$, and $c_{i,4}$, respectively. It can easily be verified that **PD** and **PM** are in mutual balance.

It remains only to show that the triples of **PM** can be partitioned into λ sections so that if two cyclic triples have an ordered pair in common, they occur in different sections. To see this, form a graph G on the vertices

$\{c_{i,j} \mid 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant 4\}$ by joining two vertices if and only if the corresponding cyclic triples have an ordered pair in common. G has maximum degree $\lambda - 1$ (in fact, it is a collection of disjoint cliques of size at most λ). Theorem 8 ensures that G has vertex colouring in λ colours. The cyclic triples can be partitioned into sections according to this vertex colouring.

PD and **PM** now satisfy the requirements of Lemma 10, which completes the embedding of **SD**.

4. Concluding Remarks

The transformation from partial triple systems to partial directed triple systems, and subsequently to partial Mendelsohn triple systems, has enabled us to obtain a small (quadratic) embedding for partial triple systems with even λ. In the case $\lambda = 2$, this can be improved to a linear embedding [9].

The technique used in proving Lemma 10 can also be applied directly to embedding partial triple systems with *any* λ. Once again, this results in a quadratic embedding, but with a worse constant than that derived here [6].

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